ANALYSIS OF THE NONLINEAR DIFFUSION SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

[1] Orzumurod Ruzikulov
[2] Zafar Rakhmonov
[1,2]National University of Uzbekistan, Uzbekistan,
E-mail: [1]r.orzumu@gmail.com, [2]zraxmonov@inbox.ru

Abstract
In this paper, we consider the solvability and unsolvability of solutions of a nonlinear diffusion system with a source and nonlinear boundary conditions in the case of slow diffusion. Establishing numerous self-similar super-solutions and sub-solutions for the nonlinear diffusion system.

Keywords: blow-up; nonlinear boundary condition; critical exponents; nonlinear diffusion system; asymptotic.

We consider the doubly degenerate parabolic equations with the source

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x}\left(\left| \frac{\partial u_i^k}{\partial x} \right|^{m-1} \frac{\partial u_i^k}{\partial x} \right) + u_i^{p_i}, x \in R_+, t > 0, i = 1,2, \qquad (1)$$

coupled through nonlinear boundary flux

$$-\left| \frac{\partial u_i^k}{\partial x} \right|^{m-1} \frac{\partial u_i^k}{\partial x} \Bigg|_{x=0} = u_{3-i}^{q_i}(0,t), t > 0, i = 1,2, \qquad (2)$$

where $m > 1, k \geq 1$ and $q_i, p_i, (i = 1,2)$ are numerical parameters. The following initial data should be taken into account

$$u_i\big|_{t=0} = u_{i0}(x), i = 1,2, \qquad (3)$$

moreover, they are expected to be continuous, non-negative and compact in R_+.

Parabolic equations with nonlinearity (1) are found in population dynamics, heat transfer [4], chemical reactions [6] and so forth [8]. The functions $u_1(x,t), u_2(x,t)$ serve as two populations' densities in terms of biology while a migration progresses or the thickness of two types of chemical reagents within a chemical process [5] and two different types of materials' temperatures during propagation. Most of all, equations (1) can be used to characterize unsteady flows in a liquid medium with a power-law dependence of shear stress on displacement velocity under polytropic conditions.

It has already been established that the local existence of weak solutions for the problem (1)-(3) is determined by the usual integration method and can be easily established as well as the comparison principle (see [3] and [7]).

Global existence and blow-up conditions of nonlinear parabolic systems are intensively studied (see [1] and references therein). In particular, there is a great interest in critical Fujita exponents for various non-linear parabolic equations in mathematical physics (see [3] and references therein).

Definition 1. The function $u(x,t)$ is seen as the inadequate remedy for the problem (1)-(3) in $\Omega=\{(0,+\infty)\times(0,T)\}$, if $0\le u_i(x,t)\in C(\Omega),(i=1,2)$ and if it complies with (1)-(3) in the sense of distribution in Ω, where $T>0$ is the longest possible time, see [9]-[10].

Theorem 1. If $0<p_i\le 1$ and $q_i\ge\dfrac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ or $p_i>1$ and $q_i\le\dfrac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ $(i=1,2)$, then, each of the solutions to (1)-(3) blows up.

Proof. To prove the theorem the subsolutions of the problem (1)-(3) have been looked for in the next form:

$$\underline{u}_i(x,t)=t^{\alpha_i}f_i(\xi_i),\ \xi_i=xt^{-\beta_i}, \tag{4}$$

where $\alpha_i=\dfrac{1}{1-p_i}$, $\beta_i=\dfrac{p_i-km}{(p_i-1)(m+1)}$, $i=1,2.$

After substitution (4) into (1)-(3) it has reached the next self-similar inequalities and boundary conditions that should be held for any $u_i(x,t)$ that treated as blow-up solutions:

$$\frac{d}{d\xi_i}\left(\left|\frac{df_i^k}{d\xi_i}\right|^{m-1}\frac{df_i^k}{d\xi_i}\right)+\beta_i\xi_i\frac{df}{d\xi_i}-\alpha_i f_i+f_i^{p_i}\ge 0, \tag{5}$$

$$-\left|\frac{\partial \underline{u}_i^k}{\partial x}\right|^{m-1}\frac{\partial \underline{u}_i^k}{\partial x}\Bigg|_{x=0} \le \underline{u}_{3-i}^{q_i}(0,t). \qquad (6)$$

Let

$$f_i(\xi_i) = A_i\left(a_i^{1+1/m} - \xi_i^{1+1/m}\right)_+^{\frac{m}{mk-1}}. \qquad (7)$$

Substitution (7) into (4), (5) lead us to the following conditions that show (5) always takes place:

$$\left(\frac{k(m+1)}{mk-1}\right)^m \left(\frac{m+1}{mk-1}\right) A_i^{mk} \ge \beta_i \frac{m+1}{mk-1} A_i,$$

$$A_i \ge \left[\beta_i\left(\frac{mk-1}{k(m+1)}\right)^m\right]^{\frac{1}{mk-1}},$$

$$f_i^{p_i} = A_i^{p_i}\left(a_i^{1+1/m} - \xi_i^{1+1/m}\right)_+^{\frac{m}{mk-1}}\left(a_i^{1+1/m} - \xi_i^{1+1/m}\right)_+^{\frac{m(p_i-1)}{mk-1}} \le$$
$$a_i^{\frac{(m+1)(p_i-1)}{mk-1}} A_i^{p_i}\left(a_i^{1+1/m} - \xi_i^{1+1/m}\right)_+^{\frac{m}{mk-1}},$$

$$A_i^{p_i} a_i^{\frac{(m+1)(p_i-1)}{mk-1}} \ge \alpha_i A_i + A_i^{mk}\left(\frac{k(m+1)}{mk-1}\right)^m,$$

By taking

$$a_i^{\frac{(m+1)(p_i-1)}{mk-1}} \ge \alpha_i A_i^{1-p_i} + A_i^{mk-p_i}\left(\frac{k(m+1)}{mk-1}\right)^m,$$

$0 < p_i \le 1$ and $q_i \ge \dfrac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ can be easily checked and ensure that A_1 and A_2 can be taken sufficient to prevent inequalities (4) and (5) are valid. Because of this, if the initial data $u_1(x,0), u_2(x,0)$ are large enough that $u_{10}(x) \ge u_1(x,0)$, $u_{20}(x) \ge u_2(x,0)$, then $\underline{u}_i(x,t), (i=1,2)$ is a subsolution to (1)-(3). According to the comparison principle, for enormous beginning data, the solutions of (1)-(3) blow up in a finite amount of time. The proof is finished.

Theorem 2. If $q_1 q_2 \le (m(k+1))^2$ and $p_i > 1$, then every solution of the problem (1)-(3) is blow-up in finite time.

Proof. Theorems 2 can be proved in the same manner as it was done in [1], [2].

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ISSN: 2835-5326 Website: econferenceseries.com

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